SOLUTION OF THE LINEARIZED COUETTE-FLOW PROBLEM IN A RAREFIED GAS BY THE INTEGRAL-DIFFUSION METHOD

A. T. Onufriev

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 47-49, 1965

In ordinary diffusion theory the transfer of properties is determined by the local gradients of the corresponding fields. As the mean free path increases, the flux density becomes an integral quantity and is determined by a neighborhood of the point under consideration of the order of a few mean free paths. In a previous article [1], the author proposed a model for a one-dimensional transfer process in linear rarefield-gas problems based on the analogy with radiative transfer. The same approach, though without directional averaging, is used in the present paper to analyze the linearized Couette flow problem. The solution obtained here has the properties of the solution obtained by more exact methods based on the solution of the Boltzmann equation [3-4].

NOMENCLATURE

p_{xy}	–	shear stress,
c	–	mean thermal velocity of molecules,
2/3 Λ	–	mean free path,
d	–	half-width of channel,
$\pm w_0$	–	plate velocity,
$\rho c \varphi$	–	"nonequilibrium" value of momentum flux density,
y	–	transverse coordinate,
γ	–	ratio of specific heats,
W	–	dimensionless velocity,
P_{xy}	–	shear stress scaled with respect to the shear stress in free-molecule flow,
Y'	–	dimensionless coordinate,
$w_1(y)$	–	velocity distribution according to Millikan's solution,
μ	–	coefficient of viscosity,
R	–	Reynolds number,
K	–	Knudsen number,

$$\mu = \frac{\Lambda \rho c}{3}, \quad R = \frac{4\rho w_0 d}{\mu}, \quad K = \frac{\Lambda}{3d} = \frac{k}{2}, \quad \gamma = \frac{c_p}{c_v}, \quad M = \frac{w}{c}, \quad W = \frac{w}{w_0}, \quad Y = \frac{y}{d}.$$

1. Consider the planar steady flow of a rarefield gas between two plates. The plates, located at $y = \pm d$, move with speeds $\pm w_0$, respectively. Wall reflection is assumed to be perfectly diffuse, and the reflected molecules have a Maxwellian distribution corresponding to wall temperature and velocity. Density, temperature, and mean free path are constant.

First, we obtain a solution by the diffusion approximation [1]. Under steady-state conditions we can write

$$p_{xy} = -{}^1/_3\, \rho c \Lambda\, \partial\varphi / \partial y = \text{const}.$$

The boundary conditions are

$$-{}^2/_3\, \Lambda\, \partial\varphi / \partial y = \varphi - w_0 \quad \text{at} \quad y = d, \qquad {}^2/_3\, \Lambda\, \partial\varphi / \partial y = \varphi + w_0 \quad \text{at} \quad y = -d.$$

The equation of transfer yields $\varphi = w$, i.e., the diffusion approximation coincides with the solution obtained by Millikan [5] on the basis of the Navier-Stokes equations with slip boundary conditions. In this solution the velocity profile is linear. The velocity slip at the wall and the shear stress are

$$\frac{\Delta w}{w_0} = \frac{k}{1+k}, \quad p_{xy} = \mu \frac{w}{d} \frac{1}{1+k} \quad \text{or} \quad C_f M = \frac{2}{(R/M)[1+\sqrt{2\pi\gamma}\, M/R]}. \tag{1.1}$$

Solutions of the kinetic equation [2, 3] show that the velocity profile in Couette flow is not linear. Near the wall there is a "Knudsen layer," whose thickness is a fraction of the mean free path, inside which the velocity profile becomes curved*.

2. To obtain an expression for the shear stress P_{xy}, we construct a model based on local equilibrium. Consider a unit area element at a point z_0, moving at the velocity of this point. We shall make the following assumptions:

(1) The flux of momentum carried by molecules that pass through this surface is determined by an integral over all space.

(2) The number of molecules that collide in the neighborhood of a point z is proportional to the particle density and inversely proportional to the mean free path Λ, and after collision these molecules have a local equilibrium distribution.

(3) The probability that a molecule, having undergone a collision, will pass through the area element under consideration is $\exp(-s/\Lambda)$, where s is the distance between z and z_0.

(4) We shall restrict the discussion to low velocities, and assume that after collision at the point z the distribution of the molecules is isotropic with respect to a system of coordinates fixed in z_0.

With these assumptions, the momentum flux per unit time and unit area is

$$-p_{xy}(y) = \int_0^{1/2\pi}\int_y^{\infty} \frac{\rho c}{2\Lambda} \exp\left(-\frac{\zeta - y}{\Lambda\cos\theta}\right) \sin\theta\, [w(\zeta) - w(y)]\, d\zeta\, d\theta -$$

$$-\int_{1/2\pi}^{\pi}\int_{-\infty}^{y} \frac{\rho c}{2\Lambda} \exp\left(-\frac{\zeta - y}{\Lambda\cos\theta}\right) \sin\theta\, [w(\zeta) - w(y)]\, d\zeta\, d\theta,$$

where θ is the angle with respect to the y axis.

The contribution of the molecules reflected from the wall is equivalent to the contribution from an infinite volume of gas located above the plane y = d and moving at the speed w_0 (and similarly in the case of the lower wall). Then

$$-p_{xy}(y) = \frac{\rho c}{2\Lambda}\int_0^{1/2\pi}\int_y^{d} \exp\left(-\frac{\zeta - y}{\Lambda\cos\theta}\right) \sin\theta\, w(\zeta)\, d\zeta\, d\theta -$$

$$-\frac{\rho c}{2\Lambda}\int_{1/2\pi}^{\pi}\int_{-d}^{y} \exp\left(-\frac{\zeta - y}{\Lambda\cos\theta}\right) \sin\theta\, w(\zeta)\, d\zeta\, d\theta +$$

$$+\frac{\rho c w_0}{2}\int_0^{1/2\pi} \cos\theta\sin\theta \exp\left(-\frac{d-y}{\Lambda\cos\theta}\right) d\theta - \frac{\rho c w_0}{2}\int_{1/2\pi}^{\pi} \cos\theta\sin\theta\exp\left(\frac{d+y}{\Lambda\cos\theta}\right) d\theta.$$

The condition $dp_{xy}/dy = 0$ leads to the integral equation for the velocity

$$W(y)\left[E_2\left(\frac{d-y}{\Lambda}\right) + E_2\left(\frac{d+y}{\Lambda}\right)\right] = E_2\left(\frac{d-y}{\Lambda}\right) - E_2\left(\frac{d+y}{\Lambda}\right) - \int_y^d \frac{W(y) - W(\zeta)}{\Lambda} \times$$

$$\times E_1\left(\frac{\zeta - y}{\Lambda}\right) d\zeta + \int_{-d}^{y} \frac{W(\zeta) - W(y)}{\Lambda} E_1\left(\frac{y-\zeta}{\Lambda}\right) d\zeta \qquad \left(E_n(t) = \int_1^{\infty} e^{-xt}\frac{dx}{x^n}\right).$$

This equation can be written in the form ($k_0 = \Lambda/d$, $z = \zeta/d$)

$$2W(y') = E_2\left(\frac{1-Y}{k_0}\right) - E_2\left(\frac{1+Y}{k_0}\right) + \int_0^1 W(Z)\left\{E_1\left(\left|\frac{Y-Z}{k_0}\right|\right) - E_1\left(\frac{Y+Z}{k_0}\right)\right\} dZ. \tag{2.1}$$

---

* The "Knudsen layer" has an analogy in the problem of radiative energy transfer between two plates held at different temperatures.

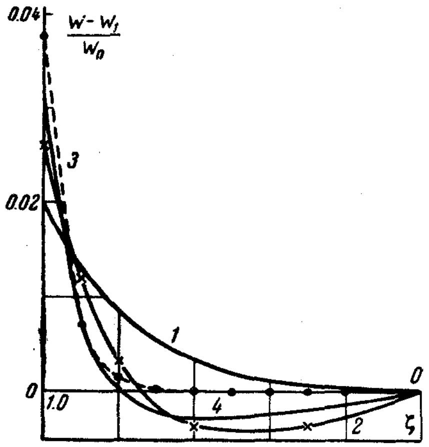
This can be compared with the equation given by Willis [2] ($\alpha = 2.7081/k_0$)

$$\pi^{1/2} W(Y) = J_0\left[\frac{\alpha(1-Y)}{2}\right] - J_0\left[\frac{\alpha(1+Y)}{2}\right] +$$

$$+ \frac{\alpha}{2}\int_0^1 W(Z)\left\{J_{-1}\left[\frac{\alpha}{2}|Y-Z|\right] - J_{-1}\left[\frac{\alpha}{2}(Y+Z)\right]\right\} dZ$$

$$\left(J_n(x) = \int_0^\infty y^n \exp\left[-\left(y^2 + \frac{x}{y}\right)\right] dy\right).$$

The values of the function $J_n(x)$ can be determined from the data in [6]. Equation (2.1) was solved approximately by reduction to a system of algebraic equations [7]. The results are shown in the table below, which gives values of the velocity $W = w(y)/w_0$ for various values of the Knudsen number. One may compare these with Willis's work [2], which gives the values W(1) = 0.851, 0.7985, 0.5038 for K = 0.120, 0.181, 0.903, respectively.



Compare also the values of the shear stress P_{xy} at various Knudsen numbers K with the corresponding values $P_{xy[2]}$ taken from [2].

$K =$	0.120	0.181	0.903	9.027	90.27
$P_{xy[2]} =$	0.189	0.261	0.623	0.936	0.992
$P_{xy} =$	0.187	0.2526	0.6008	0.92598	0.991289

A comparison of the results shows that, in the case under consideration, the proposed model yielded shear stress values within 3-4% error (wall velocity values were within 10% error).

Values of $10^4 W$

K	α	$\zeta = 0.1$	0.2	0.4	0.6	0.8	0.9	1.0
0.120	7.5	0798	1596	3196	4808	6455	7324	8395
0.181	5	0729	1460	2926	4409	5942	6763	7780
0.9027	1	0409	0818	1647	2502	3412	3913	4533
9.027	0.1	0106	0213	0424	0641	0879	1005	1154
90.27	0.01	0019	0036	0075	0112	0153	0174	0197

The solution obtained here contains a "Knudsen layer." The figure shows the difference between Millikan's solution and the velocity profile as obtained by: 1) the Mott-Smith method, 2) the Wang Chang and Uhlenbeck method, 3) the Gross and Ziering [4] method, 4) the present work. The value of the Knudsen number is K = 0.12.

REFERENCES

1. A. T. Onufriev, "A model for nonequilibrium processes in some problems in the mechanics of continua," Zhurnal prikladnoi mekhaniki i tekhnicheskoi fiziki, 1, 1963.

2. D. R. Willis, "Comparison of kinetic theory analyses of linearised Couette flow," Phys. Fluids, vol. 5, no. 2, 127-135, 1962.

3. F. S. Sherman and L. Talbot, "Comparison of kinetic theory with experiment for rarefied gases," in: Rarefied Gas Dynamics (F. M. Devienne, ed.), Proc. I International Symp. on Rarefield Gas Dynamics, Nice 1958, Pergamon Press, 1960.

4. R. E. Street, "Study of the boundary conditions in the aerodynamics of slip flow," in: Rarefied Gas Dynamics (F. M. Devienne, ed.), Proc. I International Symp. on Rarefield Gas Dynamics, Nice 1958, Pergamon Press, 1960.

5. R. A. Millikan, Phys. Rev., 21, 217-238, 1923.

6. C. T. Zahn, "Absorption coefficients for thermal neutrons," Phys. Rev., vol. 52, no. 2, 67-71, 1937.

7. L. V. Kantorovich and V. I. Krylov, Approximate Methods of Higher Analysis [in Russian], Gostekhizdat, 1950.

26 October 1964 Novosibirsk